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# Functions of generalized Besov space and their approximations by Haar Wavelet with applications in solution of Numerical Integration

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**Abstract:** In this paper, approximations of function *f* belonging to generalized Besov space by Haar Wavelet expansion have been determined. These estimators are new sharper, better and best possible in Wavelet Analysis. Numerical integrations have been calculated using Haar Wavelet expansions. It is observed that the value of integrations obtained by Haar Wavelet expansions and exact integration are almost same. This is a main achievement in Wavelet Analysis.

**Keywords:** Haar Wavelet, Haar Scaling function,Wavelet Approximation, admissibility condition, Besov space, Numerical Integration.

## I. Introduction

The theory and application of wavelets have dominated the journals of mathematics, electronics, engineering and technology. In recent year, Wavelet method provideds an powerful alternative approach for solving numerical integration. Wavelets permit the representation of several type of functions and operators. Among wavelets, Haar wavelet is simplest orthonormal wavelet with compact support which is defined by Alfred Haar in 1910. The approximation of a functions belonging to $Lip_{\alpha}(0, 1]$ , $0 < \alpha \le 1$ has been approximated by several researchers like Meyer[9], Devore[6], Debnath[4], Morlet[3], etc. In this paper, we focus our attention on generalized Besov space $f \in B_{\infty}^{\alpha,\chi}(L_{\infty}); 0<\alpha\le 1$. In an attempt to make an advance study in this direction, approximations of function $f$ belonging to generalized Besov space have been determined by the Haar wavelet method. Using the process of approximation of this paper, numerical integrations are solved. It is significant to note that the value of integration and exact integration are almost same. This is the main achievement of this paper in application of Wavelet Analysis.

## II. Definition and Preliminaries

### A. *Haar Scaling Function*

The Haar scaling function is the family of function $\varphi_{j,k}$ define By,

$$(\phi_{j,k}(t)) = 2^{j/2}\phi_H(2^j t - k), \quad j,k \in Z,$$

$$j = 0,1,2...k = 0,1,...,2^j - 1$$

$$\phi_H(t) = \begin{cases} 1, \ t \in (0,1] \\ 0, elsewhere \end{cases}$$

### B. *Haar wavelet*

The Haar wavelet, denoted by $\psi_H(t)$, is defined by

$$\psi_H(t) = \begin{cases} 1, \ t \in [0,1/2) \\ -1, t \in [1/2,1) \\ 0, elsewhere \end{cases}$$

and the Haar wavelet system is family of function $\psi_{j,k}$ defined by

$$(\psi_{j,k}(t)) = 2^{j/2}\psi_H(2^j t - k), \quad j,k \in Z,$$

$$\psi_H(t) = \begin{cases} 1, \ t \in [k/2^j, (k+0.5)/2^j) \\ -1, t \in [(k+0.5)/2^j, (k+1)/2^j) \\ 0, elsewhere \end{cases}$$

( Lepik[8])

The Haar wavelet functions are orthogonal to each other because

$$\int_0^1 \psi_{j,k}(t)\psi_{j',k'}(t)dt = \delta_{j,j'}\delta_{k,k'}$$

In which $\delta$ denotes Kronecker delta function defined by $\delta_{i,j} = \begin{cases} 1, i = j \\ 0, otherwise \end{cases}$

### C. *Wavelet Approximation*

Let $f \in L^2[0,1]$ and its Haar series be given by

$$f(t) = \sum_{k=0}^{2^N-1} \langle f, \phi_{N,k}\rangle \phi_{N,k}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} \langle f, \psi_{j,k}\rangle \psi_{j,k}(t)$$

$$= \sum_{k=0}^{2^N-1} d_{N,k}\, \phi_{N,k}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t)$$

Where $d_{N,k}$ and $c_{j,k}$ are Haar scaling and Haar coefficient respectively.

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$=(P_N f)(t)+(R_N f)(t)$

We define, $\| f \|_p = \left\{ \int_0^1 | f(t)|^p \, dt \right\}^{1/p}$ , $1 \le p < \infty$

The degree of Wavelet Approximation $E_n(f)$ of $f$ by $P_n f$ under norm $\|.\|_p$ is given by, $E_n(f)=\min_{P_n f} \| f - P_n f \|_p$

If $E_n(f) \to 0$ as $n \to \infty$ then $E_n(f)$ is called the best approximation of $f$ of order $n$. (Zygmund [1],p.115).

*D.* ***Besov Space*** $B_\infty^\alpha(L_p)$

Let $0<\alpha \le 1$ be given, for $p>1$ ,the Besov space $B_\infty^\alpha(L_p)$ is collection of all the function $f \in L_p$ such that

$$| f |_{B_\infty^\alpha(L_p)} = \sup_{0<t\le 1} \frac{\| f(x+t) - f(x) \|_p}{t^\alpha} \text{ , [Devore.et.al[7]p.844]}$$

is finite and quasi-seminorm [De-vore and Lorent[5]]. The (quasi-)norm for $B_\infty^\alpha(L_p)$ is given by

$$\| f \|_{B_\infty^\alpha(L_p)} = \| f \|_\alpha = \| f \|_p + | f |_{B_\infty^\alpha(L_p)},\ 0<\alpha \le 1.$$

*E.* ***Generalized Besov Space*** $B_\infty^{\alpha,\chi}(L_p)$

Let $\chi(t)$ be a positive monotonic function of $t$ such that $\frac{|t|^\alpha}{\chi(t)} \to 0$ as $t \to 0+$. $0<\alpha \le 1$ be given, for $p>1$ , the generalized Besov space $B_\infty^{\alpha,\chi}(L_p)$ is collection of all the function $f \in L_p$ such that

$$| f |_{B_\infty^{\alpha,\chi}(L_p)} = \sup_{0<t\le 1} \frac{\| f(x+t) - f(x) \|_p \, \chi(t)}{|t|^\alpha} \text{ and}$$

the corresponding norm is given by

$$\| f \|_{B_\infty^{\alpha,\chi}(L_p)} = \| f \|_\alpha = \| f \|_p + | f |_{B_\infty^{\alpha,\chi}(L_p)},\ 0<\alpha \le 1.$$

If $\chi(t)=c, c>0$ then $B_\infty^{\alpha,\chi}(L_p)$ class is coincides with $B_\infty^\alpha(L_p)$ class.

**Note** : It is important to note that $0<\alpha \le 1$ , $B_\infty^\alpha(L_p)$ space reduces to $H_p^\alpha$ space if $\chi(t)=1$, $\forall t \in (0,1]$.[Das et. al.[2]]

If $p \to \infty$ and $\chi(t)=1, \forall t \in (0,1]$, then $B_\infty^\alpha(L_p)$ coincides with classical Holder class $H^\alpha[0,1)$.

## III. REMARKS

Let $f(t)=c, c>0, \forall t \in [0,1)$ then

$\sup_{0<t\le 1} \frac{\| f(x+t) - f(x) \|_p}{t^\alpha} = 0$ but $f \ne 0$ Therefor, $\sup_{0<t\le 1} \frac{\| f(x+t) - f(x) \|_p}{t^\alpha} = 0$ is seminorm. Next,

$$f(t)=\begin{cases} 0,\ t \in [0,1/2) \\ -1, t=1/2 \\ 0, otherwise \end{cases}$$

Then $\left( \int_0^1 | f(t)|^p dt \right)^{1/p} = 0$ but $f(t) \ne 0$. Hence $\|.\|_p$ is not norm.

## IV. THEOREMS

In this paper we prove the following theorems:

***Theorem1:*** Let $\chi(t)$ be a positive monotonic function of $t$ such that $\frac{|t|^\alpha}{\chi(t)} \to 0$ as $t \to 0+$.If a function $f$ belongs to generalized Besov space $f \in B_\infty^{\alpha,\chi}(L_\infty)$,i.e

$$| f(x+t) - f(x) | = O\left( \frac{|t|^\alpha}{\chi(t)} \right),\ x+t, x \in [0,1), 0<\alpha \le 1$$

and wavelet series of $f$ corresponding to Haar scaling function $\phi_H = \chi_{[0,1)}$ and Haar wavelet $\psi_H = \chi_{[0,1)} - \chi_{[1/2,1)}$ is

$$f(t)=\sum_{k=0}^{2^N-1} \langle f, \phi_{N,k} \rangle \phi_{N,k}(t) + \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$$

$$=(P_N f)(t)+(R_N f)(t)$$

then the Wavelet approximation $E_n(f)$ of $f$ by $P_n f$ under norm $\|.\|_2$ is given by

$$E_N^{(1)}(f) = \| f - P_N f \|_2 = O\left( \frac{1}{\chi\left(\frac{1}{2^{N+1}}\right) 2^{N\alpha}} \right).$$

***Theorem2:***

If $| f(x+t) - f(x) | = O\left( |t|^\alpha |\sin^\beta t| \right);\ 0<\alpha-\beta<1$ i.e

$f \in B_\infty^{\alpha,\chi}(L_\infty), \chi(t)=\frac{1}{\sin^\beta t}$ then

$$E_N^{(2)}(f) = \| f - P_N f \|_2 = O\left( \frac{1}{2^{N(\alpha+\beta)}} \right).$$

***Theorem3:*** If a function $f \in B_\infty^{\alpha,\chi}(L_p)$ and $\| f \|_\alpha = \| f \|_p + | f |_{B_\infty^{\alpha,\chi}(L_p)}$ then Haar Wavelet approximation $E_n(f)$ of $f$ by $P_n f$ satisfies

$$E_N^{(3)}(f) = \| f - P_N f \|_\beta$$

$$= \| f - P_N f \|_p + \sup_{0<t\le 1} \frac{\| f(x+t) - f(x) \|_p \ \chi(t)}{|t|^\beta}$$

$$= O\left( \frac{1}{2^{(N+1)\left(\alpha-\beta-\frac{1}{p}\right)}} \right); 0 \le \beta < \alpha, \alpha > \beta + 1/p$$

## V. Proof of Theorems

***Proof of theorem1***:

$$\| f - P_N f \|_2^2 = \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2$$

$$\langle f, \psi_{j,k} \rangle = \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} f(t)\psi_{j,k}(t)dt$$

$$= \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} f(t)\psi_{j,k}(t)dt + \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} f(t)\psi_{j,k}(t)dt$$

$$= 2^{j/2} \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \left( f(t) - f\left( t + \frac{1}{2^{j+1}} \right) \right) dt$$

$$|\langle f, \psi_{j,k} \rangle| \le 2^{j/2} \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \left| f(t) - f\left( t + \frac{1}{2^{j+1}} \right) \right| dt$$

$$= O\left( 2^{j/2} \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \left( \frac{1}{2^{j+1}} \right)^\alpha \frac{1}{\psi\left( \frac{1}{2^{j+1}} \right)} dt \right); f \in B_\infty^{\alpha,\chi}(L_\infty)$$

$$= O\left( \frac{1}{2^{(j+1)(\alpha+1)-\frac{j}{2}}} \frac{1}{\psi\left( \frac{1}{2^{j+1}} \right)} \right)$$

$$(E_N^{(1)}(f))^2 = \| f - P_N f \|_2^2$$

$$\le O\left( \sum_{j=N}^{\infty} \frac{2^j}{2^{2(j+1)(\alpha+1)-j}} \frac{1}{\psi^2\left( \frac{1}{2^{j+1}} \right)} \right)$$

Now,

$$= O\left( \frac{1}{2^{2(\alpha+1)}} \sum_{j=N}^{\infty} \frac{1}{2^{2j\alpha}} \frac{1}{\psi^2\left( \frac{1}{2^{N+1}} \right)} \right)$$

$$= O\left( \left( \frac{1}{2^2(2^{2\alpha}-1)} \right) \frac{1}{2^{2N\alpha}} \frac{1}{\psi^2\left( \frac{1}{2^{N+1}} \right)} \right)$$

$$E_N^{(1)}(f) = O\left( \frac{1}{\psi\left( \frac{1}{2^{N+1}} \right) 2^{N\alpha}} \right)$$

Thus, the theorem1 is completely established.

***Proof of theorem2***:

Under the condition of theorem2, following the proof of the theorem1,

since, $|f(x+t) - f(x)| = O\left( |t|^\alpha |\sin^\beta t| \right)$

Therefore,

$$|\langle f, \psi_{j,k} \rangle| \le 2^{j/2} \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \left| f(t) - f\left( t + \frac{1}{2^{j+1}} \right) \right| dt$$

$$= O\left( 2^{j/2} \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \left( \frac{1}{2^{j+1}} \right)^\alpha \sin^\beta\left( \frac{1}{2^{j+1}} \right) dt \right)$$

$$\le O\left( \frac{1}{2^{(j+1)(\alpha+\beta+1)-\frac{j}{2}}} \right)$$

$$\| f - P_N f \|_2^2 = \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2$$

$$\le O\left( \sum_{j=N}^{\infty} \frac{2^j}{2^{2(j+1)(\alpha+\beta+1)-j}} \right)$$

$$= O\left( \frac{1}{2^{2(\alpha+\beta+1)}} \sum_{j=N}^{\infty} \frac{1}{2^{2j(\alpha+\beta)}} \right)$$

$$= O\left[ \frac{1}{2^{2(\alpha+\beta+1)}} \left( \left( \frac{1}{2^{(N-2)(\alpha+\beta)}} \right) \left( \frac{1}{2^{2(\alpha+\beta)}-1} \right) \right) \right]$$

$$= O\left[ \frac{1}{2^{2N(\alpha+\beta)+2}} \left( \frac{1}{2^{2(\alpha+\beta)}-1} \right) \right]$$

$$E_N^{(2)}(f)=O\left(\frac{1}{2^{N(\alpha+\beta)+1}\sqrt{2^{2(\alpha+\beta)}-1}}\right)$$

$$=O\left(\frac{1}{2^{N(\alpha+\beta)+1}}\right)$$

$$E_N^{(2)}(f)=O\left(\frac{1}{2^{N(\alpha+\beta)}}\right)$$

Hence, the theorem 2 has been proved.

***Proof of theorem3***:

$$c_{j,k}=\left|\left\langle f,\psi_{j,k}\right\rangle\right|$$

$$=\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} f(t)\psi_{j,k}(t)dt$$

$$=2^{j/2}\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}}\left(f(t)-f\left(t+\frac{1}{2^{j+1}}\right)\right)dt$$

$$\le 2^{j/2}\left[\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}}\left|f(t)-f\left(t+\frac{1}{2^{j+1}}\right)\right|^p dt\right]^{1/p}\left[\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}}1^q dt\right]^{1/q}$$

$$=O\left[2^{j/2}\left(\frac{1}{2^{(j+1)\alpha}}\right)\right]\left(\frac{1}{2^{(j+1)/q}}\right)$$

$$=O\left[\frac{1}{2^{j\alpha+1+\alpha+\frac{j}{2}-\frac{j}{p}-\frac{1}{p}}}\right]$$

$$\| f-P_N f\|_p^p=\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\left|\left\langle f,\psi_{j,k}\right\rangle\right|^p\|\psi_{j,k}\|_p^p$$

$$=O\left[\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\left|\left(\frac{1}{2^{j\alpha+1+\alpha+\frac{j}{2}-\frac{j}{p}-\frac{1}{p}}}\right)\right|^p\left(\frac{1}{2^{j-\frac{jp}{2}}}\right)\right]$$

$$=O\left(\sum_{j=N}^{\infty}\frac{1}{2^{j\alpha p+p+\alpha p-1-j}}\right)$$

$$=O\left(\frac{1}{2^{N\alpha p+p+\alpha p-1-N}}\right)$$

$$\| f-P_N f\|_p\le O\left(\frac{1}{2^{N\alpha+\alpha+1-\frac{1}{p}-\frac{N}{p}}}\right)$$

$$\le O\left(\frac{1}{2^{(N+1)\left(\alpha-\frac{1}{p}\right)}}\right)\le O\left(\frac{1}{2^{(N+1)\left(\alpha-\beta-\frac{1}{p}\right)}}\right)$$

Now,

$$\frac{\| f(x+t)-f(x)\|_p\,\chi(t)}{|t|^\beta}=C\frac{1}{|t|^{\beta-\alpha}}$$

$$\le C\frac{1}{2^{(N+1)(\alpha-\beta)}}$$

$$\le C\frac{1}{2^{(N+1)(\alpha-\beta-1/p)}}$$

$$\sup_{0<t\le 1}\frac{\| f(x+t)-f(x)\|_p\,\chi(t)}{|t|^\beta}=O\left(\frac{1}{2^{(N+1)(\alpha-\beta-1/p)}}\right)$$

Thus,

$$E_N^{(3)}(f)=\| f-P_N f\|_\beta$$

$$=\| f-P_N f\|_p+\sup_{0<t\le 1}\frac{\| f(x+t)-f(x)\|_p\,\chi(t)}{|t|^\beta}$$

$$=O\left(\frac{1}{2^{(N+1)\left(\alpha-\beta-\frac{1}{p}\right)}}\right);\ 0\le\beta<\alpha,\alpha-\beta>1/p$$

Hence the theorem3 has been established.

## VI. NUMERICAL CALCULATION FOR INTEGRALS

***Proposition1:***

If, $f(t)=d_{0,0}\phi_{0,0}(t)+\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}(t)\quad --(1)$ , then the estimated value of the integral is given by, $\int_0^1 f(t)dt=d_{0,0}$ .

***Proof:*** Since $\int_0^1\psi_{j,k}(t)dt=0,\forall j,k$ and $\int_0^1\phi_{0,0}(t)dt=1$

Therefore,

$$\int_0^1 f(t)dt=d_{0,0}\int_0^1\phi_{0,0}(t)dt+\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}c_{j,k}\int_0^1\psi_{j,k}(t)\,dt=d_{0,0}$$

Hence proposition 1 has been proved.

**Note.** It is quite clear that Haar approximation involves only one coefficient in the calculation of definite integral. To calculate the value of $d_{0,0}$ , we consider the nodal points

$$t_i = \frac{i+0.5}{2M}, i = 0,1,2,...,2M-1$$

The discretized form of Haar series will be

$$\sum_{i=0}^{2M-1} f(t_i) = d_{0,0} \sum_{i=0}^{2M-1} \phi_{0,0}(t_i) + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} c_{j,k} \sum_{i=0}^{2M-1} \psi_{j,k}(t_i)$$

Where, $2^J = M$

***Proposition2:***

If discretized form of equation(1) will be

$$\sum_{i=0}^{2M-1} f(t_i) = d_{0,0} \sum_{i=0}^{2M-1} \phi_{0,0}(t_i) + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} c_{j,k} \sum_{i=0}^{2M-1} \psi_{j,k}(t_i) \quad --(2)$$

then, $d_{0,0} = \frac{1}{2M} \sum_{i=0}^{2M-1} f(t_i)$.

where $t_i = \frac{i+0.5}{2M}, i = 0,1,2,...,2M-1$ and $2^J = M$.

***Proof:***

We prove the result by mathematical induction on $J$, where $2^J = M$ .For $J = 0$,we have $M = 1$

For $J = 0$, we have $M = 1$, and by equation (2), we have

$$f(t_0) = d_{0,0}\phi_{0,0}(t_0) + c_{0,0}\psi_{0,0}(t_0)$$

$$= d_{0,0} + c_{0,0} \quad --(3)$$

$$f(t_1) = d_{0,0}\phi_{0,0}(t_1) + c_{0,0}\psi_{0,0}(t_1)$$

$$= d_{0,0} - c_{0,0} \quad --(4)$$

By equations (3) and (4), $d_{0,0} = \frac{1}{2}\left(f(t_0) + f(t_1)\right)$.

Therefore, proposition is verified for $J = 0$ for corresponding $M = 1$. Next, Assume that the proposition is true for $J = n-1, n = 1,2,3,...$ ,and consider this system with $J = n$ . For $J = n$, we have $M = 2^J = 2^n$ and we have to show $d_{0,0} = \frac{1}{2*2^n} \sum_{i=0}^{2*2^n-1} f(t_i)$.

Put, $f(t_{2k}) + f(t_{2k+1}) = g(t_k)$ ; $k = 0,1,2,...,2^n-1$ . And we get

$$\sum_{k=0}^{2*2^n-1} f(t_k) = \sum_{k=0}^{2^n-1} g(t_k)$$

$$\frac{1}{2^n} \sum_{k=0}^{2*2^n-1} f(t_k) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} g(t_k)$$

$$\frac{1}{2^n} \sum_{k=0}^{2*2^n-1} f(t_k) = 2d_{0,0}$$

Therefore,

$$d_{0,0} = \frac{1}{2*2^n} \sum_{i=0}^{2^{n+1}-1} f(t_i)$$

Therefore, the proposition is true for $J = n$ i.e $M = 2^n$ whenever it is verified for $J = n-1$ i.e $M = 2^{n-1}$. Hence by mathematical induction, it is verified for all $J = 0,1,...$ .

**Note:** By proposition(1) and (2)

$$d_{0,0} = \int_0^1 f(t)dt = \frac{1}{2M} \sum_{i=0}^{2M-1} f(t_i)$$

## VII. NUMERICAL EXAMPLES

***Example 1.*** Let us consider the integral,

$$I_1 = \int_0^1 t^{1/3} dt$$

By proposition (1) and (2),

$$\int_0^1 t^{1/3} dt = \frac{1}{2M} \sum_{i=0}^{2M-1} \left(\frac{i+0.5}{2M}\right)^{1/3}$$

Let $f(t) = t^{1/3}$

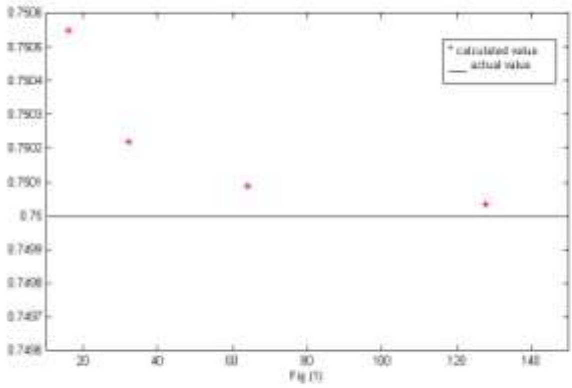
$$|f(t) - f(y)| = \left|t^{1/3} - y^{1/3}\right|$$

$$\leq |t-y|^{1/3}$$

Hence, $f \in B_{\infty}^{1/3,1}(L_{\infty})$

**Table1.** Absolute error of example 1 for different value of M.

| M | Exact value | Calculated value | Relative error |
|---|---|---|---|
| 16 | 0.750000 | 0.750549 | 5.4961E-04 |
| 32 | 0.750000 | 0.750220 | 2.2010E-04 |
| 64 | 0.750000 | 0.750087 | 8.7847E-05 |
| 128 | 0.750000 | 0.750034 | 3.4986E-05 |




graph of integral value obtained by Haar wavelet and actual value of numerical integration are shown in Fig(1).

***Example2.*** Let us consider the integral,

$$I_2 = \int_0^{\bar{1}} \cos^{1/2} t dt$$

By proposition (1) and (2),

$$\int_0^1 \cos^{1/2} t dt = \frac{1}{2M} \sum_{i=0}^{2M-1} \cos^{1/2}\left(\frac{i+0.5}{2M}\right)$$

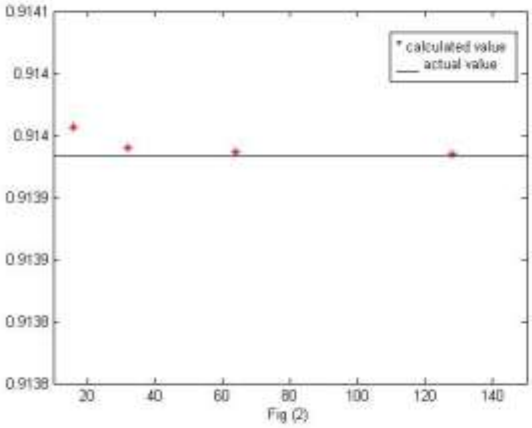
Let $f(t) = \cos^{1/2} t$

$$|f(t)-f(y)| = \left|\cos^{1/2}(t)-\cos^{1/2}(y)\right|$$
$$\leq \left|\cos(t)-\cos(y)\right|^{1/2}$$
$$= \left|2\sin\left(\frac{t+y}{2}\right)\sin\left(\frac{y-t}{2}\right)\right|^{1/2} \leq |t-y|^{1/2}$$

Hence, $f \in B_{\infty}^{1/2,1}(L_{\infty})$

**Table2.** Absolute error of example 2 for different value of M.

| M | Exact value | Calculated value | Relative error |
|---|---|---|---|
| 16 | 0.913984 | 0.914007 | 2.3289E-05 |
| 32 | 0.913984 | 0.913990 | 5.8230E-06 |
| 64 | 0.913984 | 0.913986 | 1.4575E-06 |
| 128 | 0.913984 | 0.913985 | 3.6300E-07 |




graph of integral value obtained by Haar wavelet and actual value of numerical integration are shown in Fig(2).

## VIII. CONCLUSIONS

**(1)** From the theorems1,2,3, we conclude that

(i) $E_N^{(1)}(f) = O\left(\dfrac{1}{\chi\left(\frac{1}{2^{N+1}}\right)2^{N\alpha}}\right) \to 0$ as $N \to \infty$

(ii) $E_N^{(2)}(f) = O\left(\dfrac{1}{2^{N(\alpha+\beta)}}\right) \to 0$ as $N \to \infty$

(iii) $E_N^{(3)}(f) = O\left(\dfrac{1}{2^{(N+1)\left(\alpha-\beta-\frac{1}{p}\right)}}\right) \to 0$ as $N \to \infty$

Hence all estimators $E_N^{(1)}(f), E_N^{(2)}(f), E_N^{(3)}(f)$ are sharper and best possible in wavelet analysis.

**(2)** By graph (1) and (2), it is observed that the actual values and integrations obtained by Haar wavelet of Examples (1) and (2) are almost same. This is main achievement in Wavelet analysis.

**(3)** Estimators $E_N^{(1)}(f), E_N^{(2)}(f), E_N^{(3)}(f)$ depends on $\alpha$ that assure that $f \in B_{\infty}^{\alpha}(L_p)$.

**(4)**For $f \in B_{\infty}^{\alpha,\chi}(L_p), \| f \|_p$ and

$\sup\limits_{0<t\leq 1} \dfrac{\| f(x+t)-f(x)\|_p}{t^{\alpha}}$ Are not norm but

$\| f \|_{\alpha} = \| f \|_p + \sup\limits_{0<t\leq 1} \dfrac{\| f(x+t)-f(x)\|_p}{t^{\alpha}}$ is norm and this norm considered in this paper and new $E_N^{(3)}(f)$ of a function $f$ belonging to generalized Besov space $B_{\infty}^{\alpha,\chi}(L_p)$ has been established.

This is quite new approach in Wavelet application of a function $f \in B_{\infty}^{\alpha,\chi}(L_p)$

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